Semisimple Hopf algebras

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0.1 Introduction

Semisimple Hopf algebras form a natural generalization of the notion of a group. The purpose of this paragraph is to point out in what way exactly and to give an impression of what is known about semisimple Hopf algebras. Doing this I shall restrict myself to the groundfield C. As we shall see semisimple Hopf algebras are always finite dimensional .

A Hopf algebra H over a field K is an algebra with the following linear structure maps:

1) An associative multiplication $m : H \otimes H \longrightarrow H$,

2) A unit $e : K \longrightarrow H : 1_K \longmapsto 1_H$,

3) A coassociative comultiplication $\mu : H \longrightarrow H \otimes H$,

4) A counit $\epsilon : H \longrightarrow K$,

5) A so called 'antipode' $\iota : H \longrightarrow H$,

such that the following diagrams commute:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{m\otimes id} & H \otimes H \\ {\scriptstyle id\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ H \otimes H & \xrightarrow{m} & H \end{array}$$

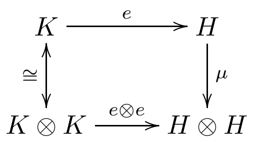
$$\begin{array}{ccc} K \otimes H & \underset{id\otimes e}{\xrightarrow{e\otimes id}} & H \otimes H \\ {\scriptstyle \cong}\updownarrow & \swarrow{\scriptstyle m} & \\ H & & \end{array}$$

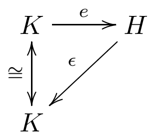
$$\begin{array}{ccc} H & \xrightarrow{\mu} & H \otimes H \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle id\otimes\mu} \\ H \otimes H & \xrightarrow{\mu\otimes id} & H \otimes H \otimes H \end{array}$$

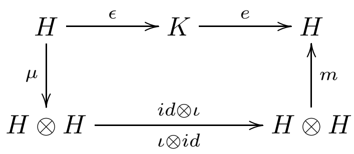
$$\begin{array}{ccc} H & \xrightarrow{\mu} & H \otimes H \\ {\scriptstyle \cong}\updownarrow & \swarrow{\scriptstyle id\otimes\epsilon,\ \epsilon\otimes id} & \\ K \otimes H & & \end{array}$$

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\mu\otimes\mu} & H \otimes H \otimes H \otimes H \\ & & \downarrow{\scriptstyle id\otimes\tau\otimes id} \\ {\scriptstyle m}\downarrow & & H \otimes H \otimes H \otimes H \\ & & \downarrow{\scriptstyle m\otimes m} \\ H & \xrightarrow{\mu} & H \otimes H \end{array}$$

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\epsilon\otimes\epsilon} & K \otimes K \\ {\scriptstyle m}\downarrow & & \updownarrow{\scriptstyle \cong} \\ H & \xrightarrow{\epsilon} & K \end{array}$$







An important example of a Hopf algebra is the group algebra $C[G]$ (G a finite

group) with the structure maps:

1) m: the ordinary group multiplication (linearly extended to $C[G]$ of course),

2) $e : 1 \longmapsto e_G$ (with e_G the unity element of G),

3) $\mu : g \longmapsto g \otimes g$,

4) $\epsilon : g \longmapsto 1$,

5) $\iota : g \longmapsto g^{-1}$.

In the sequel this Hopf algebra will be referred to as "the standard Hopf algebra".

It is easy to see that the antipode in the Hopf algebra represents the analogue of the group map which assigns to every element its inverse. For in the algebra $\mathrm{Hom}_C(C[G], C[G])$ (with pointwise addition and multiplication) the linear map $\phi : C[G] \longrightarrow C[G] : g \longmapsto e_G \quad \forall g \in G$ is the unity element. Now, as a matter of fact, you can *define* the linear map which attaches to every single groupelement its inverse as the two-sided inverse of the identity map.

You can also turn $\mathrm{Hom}_C(H, H)$ into an algebra by taking as a multiplication the convolution product, i.e. $\langle fg, c\rangle := \langle f \otimes g, \mu(c)\rangle = \sum f(c_1)g(c_2)$ $(f, g \in H, c \in H^*)$. Its unity element is given by $e \circ \epsilon$. The motivation for the definition of the antipode is obvious now.

0.2 Representation theory

An important tool in the study of groups is their representation theory. It is well known that, instead of considering the representations of G, you might as well look at the representations of $C[G]$. According to Maschke's Theorem $C[G]$ is always semisimple (if G is finite). So, if you are intending to generalize the notion of a group, the most logical thing to do is to look at semisimple Hopf algebras in general.

0.2.1 Representation theory of groups

To be able to make a good comparison between the representation theory of groups and the representation theory of Hopf algebras we will first bring back in mind some of the main results of the representation theory of groups; G is a finite group, F is a field.

• (Maschke's Theorem) The group algebra $A := F[G]$ is semisimple if and only if $\text{char}(F)|G|$.

• If A is a finite dimensional semisimple algebra over C then A is a direct sum of full matrix rings with coefficients in C.

• Let $G = C_1 = \{1\} \cup C_2 \cup \ldots \cup C_r$ be the decomposition of G into conjugacy classes. Put $c_i = \sum_{g_i \in C_i} g_i$. Then $(c_1, \ldots, c_r)$ is a base for cent $F[G]$ ([BA Proposition 5.1]).

• (Orthogonality relations) Suppose $F = C$. Let $\{\rho_1, \ldots, \rho_s\}$ be a set of representatives of the equivalence classes of irreducible representations of G over C, $\{\chi_{\rho_1}, \ldots, \chi_{\rho_s}\}$ the characters afforded by them, then

$$\sum_{g \in G} \chi_{\rho_i}(g)\chi_{\rho_j}(g) = 0 \quad (i \neq j),$$

$$\sum_{g \in G} |\chi_{\rho_i}(g)|^2 = |G|$$

• Suppose $F = C$. Let $\{\rho_1, \ldots, \rho_s\}$ be a set of representatives of the equivalence classes of irreducible representations of G over C, $\{\chi_{\rho_1}, \ldots, \chi_{\rho_s}\}$ the characters afforded by them, then for every character χ afforded by some complex representation ρ of G, we have

$$\chi = m_1\chi_{\rho_1} + m_2\chi_{\rho_2} + \ldots + m_s\chi_{\rho_s}$$

for some unique $m_i \in Z_0$.

0.2.2 Representation theory of Hopf algebras

In this section we give an exposition of the representation theory for Hopf algebras, as developed by Larson en Sweedler. As the sharp-eyed reader will notice immediately, we will assume sometimes semisimplicity, sometimes cosemisimplicity of the Hopf algebra. In the next section we will remove this seeming ambiguity in an elegant way.

Let us first introduce some frequently used module actions. Let H be a Hopf algebra, $h \in H, p \in H^*, \mu(h) = \sum h_1 \otimes h_2, a \in H$. We define:

1) $p \rightharpoonup h := \sum h_1 p(h_2)$

2) $h \leftharpoonup p := \sum p(h_1) h_2$

3) $\langle h \rightharpoonup p, a \rangle := \langle p, ah \rangle$

4) $\langle p \leftharpoonup h, a \rangle := \langle p, ha \rangle$

5) $h \rightharpoondown p := p \leftharpoonup \iota(h)$

6) $p \leftharpoondown h := \iota(h) \rightharpoonup p$

7) If (M, ρ) is a left H-comodule, we write $\rho(m) = \sum m_{-1} \otimes m_0 \in H \otimes M$, and use the notation $m \leftharpoonup p = \sum p(m_{-1}) m_0$ to describe the right (rational) H^*-module structure of M.

It is easily checked that H is a H^*-bimodule under $\leftharpoonup$ and $\rightharpoonup$ (or under $\rightharpoondown$ and $\leftharpoondown$).

Lemma 1 *H^* is a left H-Hopf module under ρ and $\rightharpoondown$*

Proof This is Theorem 5.1.2 of [Sweed] □

Furthermore we make the following notational conventions:

· If $h \in H$ we write $\langle (id \otimes \mu)\mu, h \rangle$ sometimes as $\sum h_1 \otimes h_{21} \otimes h_{22}$, but more often just as $\sum h_1 \otimes h_2 \otimes h_3$. Because of the coassociativity of the comultiplication this causes no ambiguities. (Like one also ignores the brackets in an associative multiplication).

· If (M, ρ) is a right H-comodule, $m \in M$, then we write $\rho(m) = \sum m_0 \otimes m_1$, and $\langle (\rho \otimes id), \sum m_0 \otimes m_1 \rangle = \sum m_0 \otimes m_1 \otimes m_2$

· If (M, σ) is a left H-comodule, $m \in M$, then we write $\sigma(m) = \sum m_{-1} \otimes m_0$, and $\langle (id \otimes \sigma), \sum m_{-1} \otimes m_0 \rangle = \sum m_{-2} \otimes m_{-1} \otimes m_0$. (So the m_0-part is always in M, and the $m_{\neq 0}$-parts are always in H).

To make full use of our knowledge of modules in the study of comodules, we are going to define the following important subclass of modules:

Definition 2 *Let C be a coalgebra. A left module M over C^* is called* rational *if:*

$$\forall m \in M \exists (m_i, c_i)_{\{0in\}} \in M \times C \forall c^* \in C^* : c^* \cdot m = \sum_{i=0}^{n} c^*(c_i) m_i,$$

or in other words:

There exists a right C-comodule structure on M such that:

$$c^* \cdot m = c^* \rightharpoonup m.$$

Obviously cyclic submodules of a rational module are finite dimensional . For the main part of this paper it will be even simpler.

Lemma 3 *Let C be a coalgebra. If C is finite dimensional then all C^*-modules are rational.*

Proof Choose a basis $\{e_0, \dots, e_n\} \subset C$. Let $\{e_0^*, \dots, e_n^*\}$ be the dual basis. Now let $m \in M$ be given. Define m_i $(0in)$ as follows: $m_i = e_i^* \cdot m$. Then we have for every $c^* \in C^*$: $c^* \cdot m = \sum_{i=0}^n c^*(e_i)m_i$ □

The following theorem is of fundamental importance.

Theorem 4 *If H is a Hopf algebra, $(M, \rho, \cdot)$ a right H-Hopf module and $M' = \{m \in M | \rho(m) = m \otimes 1\}$, then*

$$M' \otimes H \longrightarrow M : m' \otimes h \longmapsto m' \cdot h$$

is an isomorphism of Hopf modules.

Proof This is theorem 4.1.1 of [Sweed] □

Lemma 5 *Let H be a Hopf algebra ,$h^* \in H^*, a, b \in H$, then*

$$(h^* \rightharpoonup a)b = \sum (h^* \leftharpoondown b_2) \rightharpoonup (ab_1).$$

Proof Let $h \in H$. Since by the definition of the antipode we have:$\sum h_1 \iota(h_2) = \epsilon(h)$, it follows that

$$\sum h_1 \otimes h_2 \iota(h_3) = \sum h_1 \otimes \epsilon(h_2) = \sum h_1 \epsilon(h_2) \otimes 1 = h \otimes 1.$$

So now

$$\sum a_1 b \otimes a_2 = (m \otimes m)(id \otimes \tau \otimes id)(\mu(a) \otimes (b \otimes 1)) = \sum (m \otimes m)(id \otimes \tau \otimes id)(\mu(a) \otimes (b_1 \otimes b_2 \iota(b_3))) = \sum a_1 b_1 \otimes a_2 b_2 \iota(b_3).$$

Applying $id \otimes h^*$ to both sides we get:

$$\sum a_1 b \langle h^*, a_2 \rangle = \sum a_1 b_1 \langle h^*, a_2 b_2 \iota(b_3) \rangle.$$

By the definition of $\rightharpoonup$ and $\leftharpoondown$ the left-hand-side equals $(h^* \rightharpoonup a)b$, and the right-hand-side equals $\sum a_1 b_1 \langle h^* \leftharpoondown b_3, a_2 b_2 \rangle = \sum (h^* \leftharpoondown b_2) \rightharpoonup (ab_1)$ □

Proposition 6 [Int Proposition 2.6] *Let H be a Hopf algebra over a field k, I a nonzero right ideal in H. Then we have:*

$$H^* \rightharpoonup I = H.$$

Proof Let $g^*, h^* \in H^*, i \in I$ be given. Then we have for every $a \in H^*$:

$$\langle g^* \rightharpoonup (h^* \rightharpoonup i), a \rangle = \langle h^* \rightharpoonup i, ag^* \rangle = \langle i, ag^* h^* \rangle = \langle g^* h^* \rightharpoonup i, a \rangle.$$

So $g^* \rightharpoonup (h^* \rightharpoonup i) = g^* h^* \rightharpoonup i \in H^* \rightharpoonup I$, i.e. $H^* \rightharpoonup I$ is a left H^*-module under $\rightharpoonup$.
On the other hand we have by definition of $\rightharpoonup$ that:

$$g^* \rightharpoonup (h^* \rightharpoonup i) = \sum (h^* \rightharpoonup i)_1 g^*((h^* \rightharpoonup i)_2),$$

from which we can conclude that $\mu(H^* \rightharpoonup I) \subset (H^* \rightharpoonup I) \otimes H$, i.e. $H^* \rightharpoonup I$ is a right H-comodule.
Lemma 5 shows that $H^* \rightharpoonup I$ is a right H-module as well, so $H^* \rightharpoonup I$ is a right H-Hopf module.
Let $A := \{x \in H^* \rightharpoonup I | \mu(x) = x \otimes 1\}$. By Theorem 4 we know that: $A \otimes H \cong H^* \rightharpoonup I$ as Hopf modules, so that $A \neq \emptyset$. Now let $x \in H$ with $\mu(x) = x \otimes 1$. Then $x = \langle \epsilon \otimes id, \mu(x) \rangle = \epsilon(x) \in k$. Since $H^* \rightharpoonup I \subset H$, we have $A = k$. So the isomorphism is: $k \otimes H \longrightarrow H^* \rightharpoonup I : 1 \otimes h \longmapsto h$, and therefore $H = H^* \rightharpoonup I$ □

We are now ready to proof the theorem we promissed in the introduction.

Theorem 7 *Any semisimple (i.e. right Artinian and with zero Jacobson radical) Hopf algebra is finite dimensional .*

Proof Because H is semisimple and Ker (ϵ) is a two-sided ideal, there exists a right ideal $J \subset H$ with $H = \text{Ker}(\epsilon) \oplus J$. Now let $x \in \text{Ker}(\epsilon), y \in J$, then $yx \in \text{Ker}(\epsilon) \cap J = 0$. So for every $h \in H : yh = y((h - \epsilon(h)) + \epsilon(h)) = y\epsilon(h)$, since $h - \epsilon(h) \in \text{Ker}(\epsilon)$.
So $I := yH$ is a one dimensional right ideal of H. As we have seen in the proof of Proposition 7 $H^* \rightharpoonup I$ is a (rational!) left H^*-module under $\rightharpoonup$. It is also cyclic; generated by $\epsilon \rightharpoonup y$. So H is finite dimensional □

Definition 8 *Let H be a Hopf algebra. An element $x \in H$ is called* a left integral of H *if for every $h \in H$ we have: $hx = \epsilon(h)x$. The set $\int_L := \{x \in H : x \text{ is a left integral}\}$ is called the* space of left integrals of H.
The main reason for defining integrals becomes clear in the following crucial theorem.

Theorem 9 *Let H be a finite dimensional Hopf algebra. Then:*

$$H \text{ is semisimple } \Leftrightarrow \epsilon(\int_L) \neq 0.$$

Proof This is Theorem 5.1.8 of [Sweed] □

This theorem is a direct generalization of Maschke's Theorem . To see this, consider the standard Hopf algebra. Its space of integrals is $\int = \sum_{g \in G} g$, so in this case $\epsilon(\int) = |G| \cdot 1$.

Corollary 10 *Let H be a finite dimensional cosemisimple (see Definition 13) Hopf algebra. Then H contains a left integral λ with $\lambda(1) = 1$.*

Proof Dualize Theorem 9 □

If H is a finite dimensional Hopf algebra over a field k, K a field extension of k, then $H \otimes_k K$ is a Hopf algebra over K with the following maps:

$$m' : (H \otimes_k K) \otimes_K (H \otimes_k K) \longrightarrow (H \otimes_k K) : (a \otimes_k \alpha) \otimes_K (b \otimes_k \beta) \longmapsto ab \otimes_k \alpha\beta$$

$$\mu' : H \otimes_k K \longrightarrow (H \otimes_k K) \otimes_K (H \otimes_k K) : a \otimes_k \alpha \longmapsto \sum (a_1 \otimes_k 1) \otimes_K (a_2 \otimes_k \alpha)$$

$$\epsilon' : H \otimes_k K \longrightarrow K : a \otimes_k \alpha \longmapsto \epsilon(a)\alpha$$

$$e : K \longrightarrow H \otimes_k K : \alpha \longmapsto e(\alpha) \otimes_k 1$$

$$\iota' : H \otimes_k K : a \otimes_k \alpha \longmapsto \iota(a) \otimes_k \alpha$$

Note that in the definition of μ' we used that H is finite dimensional and therefore $(H \otimes K)^* = H^* \otimes K^* \cong H^* \otimes K$. Now it is clear that λ is a left integral in H^* if and only if $\lambda \otimes 1$ is a left integral in $H^* \otimes K$.

Using Theorem 9 this implies that H is semisimple if and only if $H \otimes_k K$ is semisimple, so generally speaking it does no harm to assume that the ground field is algebraically closed.

Proposition /Definition 11 *Let C be any coalgebra over C, (V, ρ) a simple left C-comodule. We call the unique simple subcoalgebra D with $\rho(V) \subset D \otimes V$ the* simple subcoalgebra associated with V.

Proof 1) V is a simple left C-comodule V is a simple *rational* right C^*-module V is a cyclic module V is a finite dimensional vectorspace over C.

2) Choose a basis $\{v_1, \ldots, v_n\}$ in V. Now $\rho(v_i) = \sum_j c_{ij} \otimes v_j$ for every i. We claim that the finite dimensional vectorspace D spanned by all c_{ij}'s is a coalgebra:
Since $(id \otimes \rho) \circ \rho = (\mu \otimes id) \circ \rho$,we have for every i :

$$(\mu \otimes id) \circ \rho(v_i) = (\mu \otimes id)(\sum_j c_{ij} \otimes v_j) = \sum_j (\sum c_{ij_1} \otimes c_{ij_2}) \otimes v_j$$

and

$$(id \otimes \rho) \circ \rho(v_i) = (id \otimes \rho)(\sum_j c_{ij} \otimes v_j) = \sum_j c_{ij} \otimes (\sum_k c_{jk} \otimes v_k) = \sum_k (\sum_j c_{ij} \otimes c_{jk}) \otimes v_k.$$

Hence all terms of $\mu(c_{ij})$ lie in $D \otimes D$.

3) Let $a^* \in D^*$ with $Va^* = 0$, i.e. $\sum_j a^*(c_{ij})v_j = 0 \quad \forall i$. Then $a^*(c_{ij}) = 0$ for every i, j, so $a^* = 0$, i.e. $\text{ann}(V) = 0$.

4) Suppose $0 \neq ID^*$ were a two-sided ideal. Then is xI a submodule of V for every $0 \neq x \in V$. Suppose $xI = 0$. Then define the following surjective right D^*-module homomorphism:

$$\phi : D^* \longrightarrow V : a^* \longmapsto xa^*.$$

So now $D^*/\text{Ker}(\phi) \cong V$, and $\text{ann}V = \{s \in D^* | D^*s \subset \text{Ker}(\phi)\}$. Since $I \subset \text{Ker}(\phi)$, we have $I \subset \text{ann}V$. Contradiction.
So since V is simple, we have for all $0 \neq x \in V$: $xI = V$; in particular there exists an $r_x \in I$ with $xr_x = x$.
Now choose some $0 \neq x_1 \in V, r_1 \in I$ with $x_1 r_1 = x_1$. Choose $x_2 \in V$, linearly independent of x_1, such that $x_2(1 - r_1) \neq 0$. Such x_2 exists, because otherwise $V(1 - r_1) = 0$, and thus $(1 - r_1) \in \text{ann}V$.
Call $x_2' := x_2(1 - r_1)$. Pick $r_2 \in I$ such that $x_2' r_2 = x_2'$. Then $(1 - r_1)(1 - r_2)$ annihilates the linear vector space spanned by x_1 and x_2.
Continue like this. Because V is finite dimensional this process stops after finitely many steps, so now we have found $r_1, \dots, r_t \in I$ such that $\prod_{i=1}^t (1 - r_i)$ annihilates V. But $\prod_{i=1}^t (1 - r_i) = 1 - b$ for some $b \in I$, so $\prod_{i=1}^t (1 - r_i) \neq 0$. Contradiction.
Conclusion: D^* is a simple algebra, and thus D is a simple subcoalgebra □

Obviously, two simple (sub)comodules V and W are isomorphic if and only if their associated simple subcoalgebras coincide, so we can also talk about the simple comodule V associated with the simple subcoalgebra D.

Lemma 12 [Char Lemma 1.2] *Let C be any coalgebra over C, (V, ρ) a simple left C-comodule, D the simple subcoalgebra associated with V, $\{v_1, \dots v_n\}$ a basis of V. Then there exists a basis $\{a_{ij}\}$ of D such that:*

$$\rho(v_i) = \sum_{j=1}^n a_{ij} \otimes v_j,$$

$$\mu(a_{ij}) = \sum_{k=1}^n a_{ik} \otimes akj,$$

$$\epsilon(a_{ij}) = \delta_{ij}.$$

Proof It is well known that, because D^* is simple, we have $D^* = \text{End}_C(V)$. Take $e_{ij} \in D^*$ such that $v_k \cdot e_{ij} = \delta_{ki} v_j$. Then $\{e_{ij}\}$ is a basis of D^*. Now $a_{ij} := e_{ij}^*$ is the basis we are looking for □

Lemma 13 *For any coalgebra C the following are equivalent:*

a) C is the (direct) sum of its simple subcoalgebras,
b) Every rational C^*-module is semisimple,
c) Every C-comodule is a sum of simple subcomodules V_i with $V_i \cong D_i$ for some simple subcoalgebra D_i.

Proof a) ⇔ b). This is [Sweed Theorem 14.0.1].
c) Dualize b) □

Definition 14 *Any coalgebra C satisfying the criteria of Lemma 13 is called* cosemisimple.

Corollary 15 *Let H be a finite dimensional cosemisimple Hopf algebraover C. Then H has a basis $\{a_{ij}^{(t)}\}$, $t = 1,\ldots,r$ and $i,j = 1,\ldots,d_t$ such that:*

$$\mu(a_{ij}^{(t)}) = \sum_{k=1}^{d} a_{ik}^{(t)} \otimes a_{kj}^{(t)}, \quad \text{and} \quad \epsilon(a_{ij}^{(t)}) = \delta_{ij} \qquad \square$$

Besides the direct connection between groups and algebras, another major achievement of the representation theory of groups is the introduction of characters. Especially because of the orthogonality relations they satisfy. It will be no surprise that we are going to immitate this powerful construction.

Definition 16 *Let C be a coalgebra over C, V a left C-comodule, $\{v_1,\ldots,v_n\}$ a basis of V, $\{v_1^*,\ldots,v_n^*\}$ the dual basis of V^*. Then we define the* character of V *as:*

$$\chi_V = \sum_i (id \otimes v_i^*)\rho(v_i).$$

If V is simple, then we call χ_V an irreducible *character. In that case: $\chi_V = \sum_i a_{ii}$ where $\{a_{ij}\}$ is a basis of the simple subcoalgebra associated with V as in Lemma 12.*

We call $n =$dim$V$ the *degree* of χ_V.

Corollary 17 *Let C be a cosemisimple coalgebra over C, χ_i the character of the simple comodule associated with the simple subcoalgebra D_i. Then for every comodule V we have:*

$$\chi_V = m_1\chi_1 + \ldots + m_s\chi_s$$

for some unique $m_i \in Z_0$.

Proof By Lemma 13 we have: $V \cong \oplus_{i=1}^{s}(V_i)^{m_i}$ for some unique $m_i \in Z_0$, V_i the simple subcomodule associated with the simple subcoalgebra D_i. Furthermore it is easy to see that for V,W simple comodules we have: $\chi_{V\oplus W} = \chi_V + \chi_W$ $\square$

Let H be a Hopf algebra with antipode ι, and let V,W be finite dimensional right H-comodules. Then $V \otimes W$ is a right H-comodule with comodule map: $\rho^R : (v,w) \longmapsto \sum v_0 \otimes w_0 \otimes v_1 w_1$, and it is a left H-comodule with comodule map: $\rho^L : (v,w) \longmapsto \sum \iota(v_1)\iota(w_1) \otimes v_0 \otimes w_0$.

Let $\{v_0,\ldots,v_n\}$ be a basis of V; $\psi : V \longrightarrow V \otimes H : v_j \longmapsto \sum_i v_i \otimes a_{ij}$ the comodule map. Then V^* is a left H-comodule with comodule map: $v_j^* \longmapsto \sum_i a_{ji} \otimes v_i^*$ (with the v_j^*s of course being the dual basis).

V^* is also a right H-comodule via $v_j^* \longmapsto \sum_i v_i^* \otimes \iota(a_{ji})$.

Finally: $\mathrm{Hom}_C(V,W) \cong W \otimes V^*$ by $(w \otimes v^*)(v) := v^*(v)w \ \forall v \in V$

Theorem 18 [Char Theorem 2.7] *(Orthogonality relations) Let H be a cosemisimple Hopf algebra over C, and let $\lambda \in H^*$ satisfy $\lambda a^* = a^*(1)\lambda \ \forall a^* \in H$ and $\lambda(1) = 1$. If V,W are simple nonisomorphic right H-comodules then*

$$\lambda(\chi_W \iota(\chi_V)) = 0,$$
$$\lambda(\chi_V \iota(\chi_V)) = 1.$$

Proof For every $v^* \in V^*$, and $v \in V$ we have:

$$\sum v_0^*(v)v_{-1}^* = \sum v^*(v_0)v_1.$$

Now let $f \in \mathrm{Hom}(V,W)$. Write $f = \sum w \otimes v^*$. Then

$$\rho^L(f)(v) = \rho^L(\sum w \otimes v^*)(v) = (\sum \iota(w_1)v^*_{-1} \otimes w_0 \otimes v^*_0)(v) =$$

$$\sum \iota(w_1)v^*_{-1} \otimes v^*_0(v)w_0 = \sum \iota(w_1)v_1 \otimes v^*(v_0)w_0 =$$

$$\sum \iota(f(v_0)_1)v_1 \otimes f(v_0)_0.$$

Let V, W be simple right C-comodules, $\rho^L : W \otimes V^* \longrightarrow (H \otimes W \otimes V^*) : w \otimes v^* \longmapsto \sum \iota(w_1)v^*_1 \otimes w_0 \otimes v^*_0$ the comodule map as described above. Because for every $a^* \in H^* : (f \cdot \lambda) \cdot a^* = a^*(1)(f \cdot \lambda)$, we know that: $\rho^L(f \cdot \lambda) = 1 \otimes f \cdot \lambda$. Combining these two results we get:

$$\begin{array}{ccc}
\sum \iota(f \cdot \lambda(v_0)_1)v_1 \otimes f \cdot \lambda(v_0)_0 & = & 1 \otimes f \cdot \lambda(v) \\
\downarrow {\scriptstyle id \otimes \rho^R_W} & & \downarrow {\scriptstyle id \otimes \rho^R_W} \\
\sum \iota(f \cdot \lambda(v_0)_2)v_1 \otimes f \cdot \lambda(v_0)_0 \otimes f \cdot \lambda(v_0)_1 & = & \sum 1 \otimes f \cdot \lambda(v)_0 \otimes f \cdot \lambda(v)_1 \\
\downarrow {\scriptstyle (m \otimes id)\circ(\tau \otimes id)\circ(id \otimes \tau)} & & \downarrow {\scriptstyle (m \otimes id)\circ(\tau \otimes id)\circ(id \otimes \tau)} \\
\sum f \cdot \lambda(v_0)_1 \iota(f \cdot \lambda(v_0)_2)v_1 \otimes f \cdot \lambda(v_0)_0 & = & \sum f \cdot \lambda(v)_1 \otimes f \cdot \lambda(v)_0
\end{array}$$

The left-hand-side equals $\sum \epsilon(f \cdot \lambda(v_0)_1)v_1 \otimes f \cdot \lambda(v_0)_0$, so:

$$\sum v_1 \otimes f \cdot \lambda(v_0) = \sum f \cdot \lambda(v)_1 \otimes f \cdot \lambda(v)_0$$

i.e. $f \cdot \lambda$ is a comodule map. By Schur's Lemma this implies that $f \cdot \lambda = 0$ when VW, and $f \cdot \lambda = z_{f \cdot \lambda} \cdot 1 \in C$ when $V \cong W$.
Now let V and W be nonisomorphic,$\{v_i\}$ and $\{w_k\}$ bases of V and W, $\{v^*_i\}$ and $\{w^*_k\}$ the dual bases of V^* and W^*. Let $\{a_{ij}\}$ and $\{b_{kl}\}$ the sets of elements in H such that $\rho^R_V(v_k) = \sum_l v_l \otimes b_{kl}$ and $\rho^R_W(w_i) = \sum_j w_j \otimes a_{ij}$. Consider $\mathrm{Hom}(V,W) \ni f = w_n \otimes v^*_m$. As shown above we have:

$$0 = (w_n \otimes v^*_m) \cdot \lambda = \sum_{i,j} \lambda(a_{ni}\iota(b_{jm}))w_i \otimes v^*_j.$$

Where the second equality is obtained by direct computation. So

$$\lambda(a_{ni}\iota(b_{jm})) = 0 \qquad \forall i,j,m,n \quad (1)$$

Similarly, if we take $\mathrm{End}(V) \ni g = v_n \otimes v^*_m$, we obtain that

$$\lambda(b_{ni}\iota(b_{jm})) = \delta_{nm} z_{v_i \otimes v^*_j} \quad (2)$$

Equation (2) gives:

$$1 = \lambda(\epsilon(b_{11})1) = \lambda(\sum_m b_{1m}\iota(b_{m1})) = \sum_m z_{v_m \otimes v^*_m} \quad (3)$$

Now equation (1) gives:

$$\lambda(\chi_W \iota(\chi_V)) = \lambda(\sum_{m,n} a_{mm}\iota(b_{nn})) = 0$$

Finally equations (2) and (3) give:

$$\lambda(\chi_V \iota(\chi_V)) = \lambda(\sum_{m,n} b_{mm}\iota(b_{nn})) = \sum_m z_{v_m \otimes v^*_m} = 1 \qquad \square$$

Definition *Let H be a Hopf algebra. We define $i(H)$ as the subalgebra consisting of all cocommutative elements.*

Theorem 19 *Let H be a cosemisimple Hopf algebraover C. The set of characters of the simple comodules of H is a basis of $i(H)$. If C is a simple subcoalgebra of H, then $C \cap i(H)$ is one-dimensional, and consists of all scalar multiples of the character of a simple comodule associated with C.*

Proof This is Lemma 3.1 of [Char] □

0.3 The two representation theories match exactly

0.3.1 Cosemisimple if and only if semisimple

¿From now on we shall use freely the results of [Sweed Proposition 4.0.1].

With the idea of semisimple Hopf algebras representing a generalization of group algebras in mind it is desirable that $\iota^2 = id$. Hence, it might be worth to pay some special attention to the square of the antipode. Doing this, its study turns out to bring a lot of structure to the surface.

Definition 20 Let H be a Hopf algebra. We define the linear map $L : H^* \longrightarrow \mathrm{End}(H^*)$ through

$$L(p)(q) := pq \qquad\qquad (p, q \in H^*).$$

We can now state the following proposition:

Proposition 21 [Cos Proposition 1.1] *Let H be a finite dimensional Hopf algebra over C, λ a nonzero right integral in H^*.*

a) There exists a right integral Λ in H with $\lambda(\Lambda) = 1$. (Consequently for every right integral Λ in H we have $\lambda(\Lambda) \neq 0$).

b) $(a \rightharpoonup \lambda)p = (a \leftharpoonup p) \rightharpoonup \lambda$ for every $a \in H$ and $p \in H^*$.

c) $L(a \rightharpoonup \lambda) \circ f^* = \sum f(a_1) \otimes (a_2 \rightharpoonup \lambda)$ for $a \in H$ and $f \in End(H)$, identifying $H \otimes H^*$ with $End(H^*)$ by $(a \otimes a^*)(b^*) = a(b^*)a^*$.

Proof a) Since $(H^*, m^*, \leftharpoonup)$ is a right H-Hopf module, and $\int_R = \{h^* \in H^* | m^*(h^*) = 1 \otimes h^*\}$, we obtain by Theorem 4 that $H \longrightarrow H^* : h \longmapsto (\lambda \leftharpoonup h)$ is an isomorphism of Hopf modules. From this we can conclude:
· $hg = h\epsilon(g)\ \forall g \Leftrightarrow \lambda \leftharpoonup hg = \lambda \leftharpoonup h\epsilon(g)\ \forall g \Leftrightarrow \forall p : (\lambda \leftharpoonup hg)(p) = (\lambda \leftharpoonup h\epsilon(g))(p)$,
· there exists a $\Lambda \in H$ such that $\lambda \leftharpoonup \Lambda = \epsilon$.
And now we have for every $g, p \in H$: $(\lambda \leftharpoonup \Lambda g)(p) = \lambda(\Lambda gp) = (\lambda \leftharpoonup \Lambda)(gp) = \epsilon(gp) = \epsilon(g)\epsilon(p) = \epsilon(g)(\lambda \leftharpoonup \Lambda)(p) = (\lambda \leftharpoonup \Lambda\epsilon(g))(p)$, which proves that this Λ is a right integral.
b) Analoguous to the proof of Lemma 5, taking into account the fact that $\rho^L(\lambda) = 1 \otimes \lambda$.
c) Suppose $a \in H$, $q \in H^*$, and $f \in \mathrm{End}(H)$. Now:

$$L(a \rightharpoonup \lambda) \circ f^*(q) = (a \rightharpoonup \lambda)(q \circ f) \overset{*}{=} (a \leftharpoonup (q \circ f)) \rightharpoonup \lambda = \sum q(f(a_1))a_2 \rightharpoonup \lambda \overset{**}{=} (\sum f(a_1) \otimes (a_2 \rightharpoonup \lambda))(q)$$

for every $q \in H^*$.

* According to b)
** Here we identify $H \otimes H^*$ with $\mathrm{End}(H^*)$ □

Definition 22 Let $f \in \mathrm{End}(H)$. We define $\Lambda_f \in H$ by

$$\Lambda_f(p) := \mathrm{Tr}(L(p) \circ f^*)$$

where $p \in H^*$.

Note that $\epsilon(\Lambda_f) = \mathrm{Tr}(f^*) = \mathrm{Tr}(f)$

Lemma 23 [Cos Lemma 2.2] *Let H be a Hopf algebra over C, $\lambda \in H^*$ a nonzero right integral.Then*

$$a \rightharpoonup \lambda(\Lambda_f) = Tr(L(a \rightharpoonup \lambda) \circ f^*) = \sum \lambda(\iota(a_2)f(a_1))$$

for $a \in H$ and $f \in End(H)$.

Proof By Proposition 21 we have: $L(a \rightharpoondown \lambda) \circ f^* = \sum f(a_1) \otimes (a_2 \rightharpoondown \lambda)$. Under the identification made, we have: $\mathrm{Tr}(a \otimes a^*) = a^*(a)$. So $\mathrm{Tr}(L(a \rightharpoondown \lambda) \circ f^*) = \sum a_2 \rightharpoondown \lambda(f(a_1))$ □

Now let $\widetilde{\Lambda} := \Lambda_{\iota^2}$

Lemma 24 *$a \rightharpoondown \lambda(\widetilde{\Lambda}) = \epsilon(a)\lambda(1)$ for $a \in H$.*

Proof Using Lemma 23 we get: $a \rightharpoondown \lambda(\widetilde{\Lambda}) = \sum \lambda(\iota(a_2)\iota^2(a_1)) = \sum \lambda \circ \iota(\iota(a_1)a_2) = \lambda \circ (\epsilon(a)1) = \epsilon(a)\lambda(1)$ □

Note that by definition: $\epsilon(\widetilde{\Lambda}) = \mathrm{Tr}(\iota^2)$.

After having done the groundwork, we can now proof the result which it was all good for:

Theorem 25 [Cos Theorem 2.5] *a) Let $\lambda \in H^*$ and $\Lambda \in H$ be right integrals satisfying $\lambda(\Lambda) = 1$. Then $Tr(\iota^2) = \epsilon(\Lambda)\lambda(1)$.*

b) H and H^* are semisimple if and only if $\mathrm{Tr}(\iota^2) \neq 0$.

Proof a) $\iota^{-1}(\Lambda) \rightharpoondown \lambda = \epsilon$, since for every $a \in H$:
$\iota^{-1}(\Lambda) \rightharpoondown \lambda(a) = \lambda(\Lambda a) = \epsilon(a)\lambda(\Lambda) = \epsilon(a)$. Therefore by lemma 24 and the remark following it, we have that $\mathrm{Tr}(\iota^2) = \epsilon(\widetilde{\Lambda}) = \iota^{-1}(\Lambda) \rightharpoondown \lambda(\widetilde{\Lambda}) = \epsilon(\iota^{-1}(\Lambda))\lambda(1) = \epsilon(\Lambda)\lambda(1)$.
b) By Proposition 21a) there exist right integrals $\Lambda \in H$ and $\lambda \in H^*$ satisfying $\lambda(\Lambda) = 1$. From Theorem 9 we know that H is semisimple if and only if $\epsilon(\Lambda) \neq 0$, and H^* is semisimple if and only if $\lambda(1) \neq 0$, so part b) follows from part a) □

But we can also say more about ι^2 itself. Namely, if H is finite dimensional , then ι^2 is a coalgebra automorphism, so obviously it sends simple subcoalgebras to simple subcoalgebras of the same dimension. If in addition H is cosemisimple , then the situation is even more restrictive.

Theorem 26 [Char Theorem 3.3] *Let H be a finite dimensional cosemisimple Hopf algebra over C. Then for each simple subcoalgebra C we have: $\iota^2(C) = C$.*

Proof By [Sweed Corollary 5.1.6] ι as well as ι^* are bijective. Therefore: for every $a^* \in H^*$ there exists a $b \in H^*$ with $a^* = \iota^*(b^*)$, so $\iota^*(\lambda)a^* = \iota^*(\lambda)\iota^*(b^*) = \iota^*(b^*\lambda) = b^*(1)\iota^*(\lambda) = a^*(1)\iota^*(\lambda)$ where $\lambda \in H^*$ is a left integral. Furthermore, if we choose λ such that $\lambda(1) = 1$, then $\iota^*(\lambda)(1) = \lambda(\iota(1)) = \lambda(1) = 1$, and so

$$\iota^*(\lambda)\lambda = \epsilon_H(\iota^*(\lambda))\lambda = \iota^*(\lambda)(1)\lambda = \lambda$$

$$\iota^*(\lambda)\lambda = \lambda(1)\iota^*(\lambda) = \iota^*(\lambda).$$

Now since ι is bijective and a coalgebra antiendomorphism, $\iota(C)$ is a simple subcoalgebra. Let χ_C be the character associated with C. Then $\iota(\chi_C)$ is the character associated with $\iota(C)$, and $\iota^2(\chi_C)$ is the character associated with $\iota^2(C)$. (This is obvious: suppose $\{a_{ij}\}$ is a base of C, then $\chi_C = \sum_i a_{ii}$. Because ι is a coalgebra antihomomorphism, $\{\iota(a_{ij})\}$ is a base of $\iota(C)$. So $\chi_{\iota(C)} = \sum_i \iota(a_{ii}) = \iota(\sum_i a_{ii}) = \iota(\chi_C)$). So now we find: $\iota^2(\chi_C) \stackrel{*}{=} \sum \lambda(\iota^2(\chi_C)\iota(\chi))\chi = \sum \iota^*(\lambda)(\chi\iota(\chi_C))\chi = \sum \lambda(\chi\iota(\chi_C))\chi = \chi_C$.
By Theorem 19 this means: $C \cap i(H) = \iota^2(C) \cap i(H)$. Since C and $\iota^2(C)$ are simple subcoalgebras this implies that $\iota^2(C) = C$ □

* By Theorem 18

Theorem 27 [Cos Theorem 3.3] *Let H be a finite dimensional cosemisimple Hopf algebra with antipode ι*

over C*. Then* H *is semisimple*

Proof By [Char Corollary 5.6] $(\iota^2)^n = id$ for some $n0$, which implies by elementary linear algebra that ι^2 is diagonizable.
Now let $C \subset H$ be a simple subcoalgebra. Choose a basis $\{m_1 \ldots, m_s\}$ of C such that $\iota^2(m_i) = \lambda_i m_i$ with $\lambda_i \in C$.(Here we use Theorem 26). Obviously $\lambda_i^n = 1$, so $\lambda_i^{-1} = \overline{\lambda_i}$ (its complex conjugate). We have:

$$\mu(\lambda_i m_i) = \lambda_i \sum_j c_{ij} \otimes m_j \quad \text{for certain } c_{ij} \in C$$

and:

$$\mu(\lambda_i m_i) = \mu(\iota^2(m_i)) \stackrel{*}{=} \sum_j \iota^2(c_{ij}) \otimes \iota^2(m_j) = \sum \lambda_j \iota^2(c_{ij}) \otimes m_j.$$

* since ι^2 is a coalgebra isomorphism.
So $\iota^2(c_{ij}) = \lambda_i \lambda_j^{-1} c_{ij} = \lambda_i \overline{\lambda_j} c_{ij}$ for every i, j.
Furthermore, it is clear that the linear vectorspace spanned by the c_{ij}'s is a subcoalgebra of C. Since C is simple this means that $\{c_{ij}\}$ is a basis of C. From this we can conclude that:

$$\mathrm{Tr}(\iota^2|_C) = (\sum_i \lambda_i)(\sum_j \overline{\lambda_j}) = (\sum_i \lambda_i)\overline{(\sum_j \lambda_j)} = |\sum_i \lambda_i|^2 \in R_0.$$

Because all of this holds for every simple subcoalgebra C we see that:

$$\mathrm{Tr}(\iota^2) = 1 + \sum_{C \neq C \cdot 1} \mathrm{Tr}(\iota^2|_C) \neq 0.$$

By Theorem 25 we can therefore conclude that H is semisimple □

As is easily checked for any coalgebra C the map: $\phi : \{\text{simple subcoalgebras of } C\} \longrightarrow \{\text{maximal ideals of } C^*\} : D \longmapsto D^\perp$ is a bijection.

Lemma 28 *Let* A *be a finite dimensional algebra. Then the Jacobson radical* J *is the intersection of all maximal two-sided ideals.*

Note that this is not generally true! The first Weyl algebra, for instance, is a counterexample.

Proof ($\bullet\ J = 0$) In this case $A \cong A_1 \oplus \ldots \oplus A_s$ with the A_is simple subalgebras. Now for every $i : \oplus_{j \neq i} A_j$ is a maximal two-sided ideal. So now $\bigcap_{j=1}^s \oplus_{i \neq j} A_i = 0 = J$.
($\bullet\ J \neq 0$) The only thing left to be proved now is that J is contained in every maximal ideal.
Let $I \subset A$ be an arbitrary maximal ideal, and consider the subcoalgebra (of A^*) $D := I^\perp$. Then $(D, \leftharpoonup)$ is a right A-module. Choose a simple submodule E of D, then $\mathrm{ann}E \supset \mathrm{ann}D \supset I^{\perp\perp} \supset I$. Because $\mathrm{ann}E$ is a two-sided ideal and I is maximal, this implies that $\mathrm{ann}E = I$.
So now $J = \bigcap\{\mathrm{ann}M | M \text{ is a simple right } A\text{-module}\} \subset I$ □

Corollary 29 *Let* H *be a semisimple Hopf algebra over* C*. Then* H *is cosemisimple .*

Proof Define $H_0^* := \{\bigoplus_i D_i \subset H^* | D_i \text{ is a simple subcoalgebra of } H^*\}$, i.e. H_0^* is the biggest cosemisimple subcoalgebra of H^*. According to Lemma 28 we have: $0 = J = (H_0^*)^\perp$, so that $H^* = H_0^*$. But then H^* is semisimple by Theorem 27, which implies in its turn that $H^{**} = H$ is cosemisimple □

0.3.2 Kaplansky's conjecture: $\iota^2 = id$

It has long been an open question wether or not the antipode of a semisimple Hopf algebra over Cis always of order 2. The answer to this question turns out to fit surprisingly well in our intuitive idea of a semisimle Hopf algebra.
For convenience we call $x := \Lambda_{id}$. Then $a \rightharpoondown \lambda(x) = \sum \lambda(\iota(a_2)a_1)\ \forall a \in H$ by Lemma 23.

Lemma 30 [Cos Lemma 1.3] *Let $\beta : V \otimes V \longrightarrow C$ be a non-singular bilinear form on the finite dimensional vector space V over C, and let $S, T \in End_C(V)$.*

If $\beta(S(u), v) = \beta(u, T(v))$ for every $u, v \in V$, then $\mathrm{Tr}(S) = \mathrm{Tr}(T)$.

Proof Define $\beta_l; V \longrightarrow V^* : z \longmapsto \beta(z, \cdot)$. Because β is non-singular β_l is a bijection. let $y_1 \dots y_n$ be a basis for V and let $\beta_l(x_1), \dots, \beta_l(x_n)$ be the dual basis (with some severely abuse of notation). Then $\beta(x_i, y_j) = \beta_l(x_i)(y_j) = \delta_{ij}$. Under the usual identification of $V \otimes V^*$ with $\mathrm{End}_C(V)$ given by $v \otimes v^*(w) = v^*(w)v$ we obtain that $S = \sum_i S(x_i) \otimes \beta_r(y_i)$ and $T = \sum_i T(y_i) \otimes \beta_l(x_i)$. With $\mathrm{Tr}(v \otimes v^*) = v^*(v)$ we compute $\mathrm{Tr}(S) = \sum_i \beta_r(y_i)(S(x_i)) = \sum_i \beta_l(x_i)(T(y_i)) = \mathrm{Tr}(T)$ □

Proposition 31 [Sem Proposition 2] *Let H be a finite dimensional Hopf algebra over C. Then*

a) $\dim(H) \cdot 1 = \epsilon(x)$

b)*x is cocommutative*

c) *$t(x) = x$ for any coalgebra automorphism t* of *H*

d) *$cx = \epsilon(c)x$* if *$c \in H$ is cocommutative*

e) $x^2 = \epsilon(x)x$

f) *$p(x) = p(1)$ if $p \in H^*$ is a left integral*

Proof a) Follows from the definition of x
b) Let $p, q \in H^*$. Then $\mathrm{Tr}(L(p) \circ L(q)) = \mathrm{Tr}(L(q) \circ L(p))$. So $pq(x) = \mathrm{Tr}(L(pq)) = \mathrm{Tr}(L(p) \circ L(q)) = \mathrm{Tr}(L(q) \circ L(p)$ shows that $pq(x) = qp(x)$. Since $pq(x) = \sum p(x_1)q(x_2)$ it now follows that x is cocommutative.
c) Because t is a coalgebra automorphism of H it follows that $T = t^*$ is an algebra automorphism of H^*. Thus for $p \in H^*$ we have $p(t(x)) = T(p)(x) = \mathrm{Tr}(L(T(p))) = \mathrm{Tr}(T \circ L(p) \circ T^{-1}) = \mathrm{Tr}(L(p)) = p(x)$ which implies that $t(x) = x$.
d) Suppose $c \in H$ is cocommutative and let $\lambda \in H^*$ be a right integral. Since ι is a coalgebra antihomomorphism, ι^{-1} is one also. Therefore $d = \iota^{-1}(c)$ is cocommutative. This means that $\sum \iota(d_2)d_1 = \sum \iota(d_1)d_2 = \epsilon(d) \cdot 1$, so $\sum \iota(d_2)d_1 = \epsilon(c) \cdot 1$, since $\epsilon(d) = \epsilon(c)$. We can now calculate: $a \rightharpoondown \lambda(cx) = \iota^{-1}(c)a \rightharpoondown \lambda(x) = da \rightharpoondown \lambda(x) \stackrel{*}{=} \sum \lambda(\iota(d_2a_2)d_1a_1) = \sum \lambda(\iota(a_2)\iota(d_2)d_1a_1) = \sum \lambda(\epsilon(c)\iota(a_2)a_1) = a \rightharpoondown \lambda(\epsilon(c)x)$ for all $a \in H$. Because by Theorem 4 $H \rightharpoondown \lambda = H^*$ we conclude: $cx = \epsilon(c)x$

* By Lemma 23
e) Trivial.
f) By Lemma 30 we have that $\mathrm{Tr}(L(q)) = \mathrm{Tr}(R(q))$ for any $q \in H^*$. Since p is a left integral, we have that $\mathrm{Tr}(R(p)) = p(1)$. Therefore $p(x) = \mathrm{Tr}(L(p)) = \mathrm{Tr}(R(p)) = p(1)$ □

Lemma 32 a) $\mathrm{Tr}(\iota^2) = \mathrm{Tr}(L(x) \circ \iota^2)$

b) $\mathrm{Tr}(\iota^2) = (\dim H)\mathrm{Tr}(\iota^2|_{xH})$

Proof a) It is easy to see that $\widetilde{\Lambda}$ as defined above Lemma 24 is a left integral. Namely, let $\lambda \in H^*$ be a nonzero right integral, $b \in H$, then

$$a \rightharpoonup \lambda(b\,\widetilde{\Lambda}) = \iota^{-1}(b)a \rightharpoonup \lambda(\widetilde{\Lambda}) \stackrel{*}{=} \epsilon(\iota^{-1}(b)a)\lambda(1) = \epsilon(b)\epsilon(a)\lambda(1) = a \rightharpoonup \lambda(\epsilon(b)\,\widetilde{\Lambda})$$

for all $a \in H^*$ so $b\,\widetilde{\Lambda} = \epsilon(b)\,\widetilde{\Lambda}$.

* By Lemma 24

Now by Proposition 31f) we compute: $\mathrm{Tr}(\iota^2) = \widetilde{\Lambda}\,(1) = \widetilde{\Lambda}\,(x) = \mathrm{Tr}(L(x) \circ \iota^2)$.

b)By Proposition 31c) we have that $\iota^2(x) = x$, since ι^2 is a coalgebra automorphism. Therefore the right ideal xH is invariant under ι^2, since ι^2 is an algebra automorphism. As $\epsilon(x) = (\mathrm{dim}H)\cdot 1$ by Proposition 31a) we have to establish:

$$\mathrm{Tr}(\iota^2) = \epsilon(x)\mathrm{Tr}(\iota^2|_{xH}).$$

Since $x^2 = \epsilon(x)x$ (Proposition 31e)), it follows that $L(x)^2 = \epsilon(x)L(x)$. Since $\iota^2(x) = x$, it follows that $L(x)$ and ι^2 commute. We consider two cases:

Case 1: $\epsilon(x) = 0$

In this case $L(x)^2 = 0$, so $(L(x) \circ \iota^2)^2 = 0$ as well, since $L(x)$ and ι^2 commute. Now by a) we have: $\mathrm{Tr}(\iota^2) = \mathrm{Tr}(L(x) \circ \iota^2) = 0$. So b) holds in this case.

Case 2: $\epsilon(x) \neq 0$

In this case $e := x/\epsilon(x)$ is an idempotent, and therefore $E := L(e)$ is an idempotent operator. Since E and ι^2 commute, it follows that:

$$\mathrm{Tr}(L(e) \circ \iota^2) = \mathrm{Tr}(\iota^2 \circ E) = \mathrm{Tr}(\iota^2|_{\mathrm{Im}(E)}) = \mathrm{Tr}(\iota^2|_{xH}).$$

Using a) we find:

$$\mathrm{Tr}(\iota^2) = \mathrm{Tr}(L(x) \circ \iota^2) = \epsilon(x)\mathrm{Tr}(L(e) \circ \iota^2) = \epsilon(x)\mathrm{Tr}(\iota^2|_{xH}) \qquad \square$$

Theorem 33 [Sem Theorem 4] *Let H be a semisimple Hopf algebra over C. Then $\iota^2 = id$.*

Proof Since H and H^* are both semisimple , they are both unimodular, so $\iota^4 = id$ by [Cos Corollary 2.7]. Therefore ι^2 is diagonizable with its eigenvalues among ± 1. Let

$$n_i := \mathrm{dim}(a \in H | \iota^2(a) = ia) \text{ for } i = \pm 1.$$

Then $\mathrm{Tr}(\iota^2) = (n_1 - n_{-1})1$ and $\mathrm{dim}H = n_1 + n_{-1}$. We will prove the theorem by showing that $n_{-1} = 0$. From Theorem 25 we know that $\mathrm{Tr}(\iota^2) \neq 0$. Therefore $0 < |n_1 - n_{-1}|n_1 + n_{-1} = \mathrm{dim}H$. To prove the theorem , it suffices to show that $n_1 - n_{-1}$ is an integral multiple of $\mathrm{dim}H$. For then $|n_1 - n_{-1}| = n_1 + n_{-1}$ which implies that $n_1 = 0$ or $n_{-1} = 0$. Since $\iota^2(1) = 1$ we know that $n_1 > 0$ and hence $n_{-1} = 0$.
We now show that $n_1 - n_{-1}$ is an integral multiple of $\mathrm{dim}H$. First suppose that $xH = H$. Then $\mathrm{dim}H = 1$ by Lemma 32b), and we are done.

Now suppose that $xH \neq H$. We know that $\mathrm{Tr}(\iota^2|_{xH}) = m$ for some integer m satisfying $|m|\mathrm{dim}xH$ (since ι^2 is a diagonalmatrix with as entries only +1 and -1) So in this case $|m|\mathrm{dim}H - 1$ (since $xH \neq H$). Set $d = (n_1 - n_{-1}) - m\mathrm{dim}H$. Then $d = 0$ by Lemma 32b), or equivalently: $n_1 - n_{-1} = \mathrm{dim}H$ $\square$

The reverse implication is already stated in Theorem 25.

0.4 The algebra $\mathbf{C} \oplus M_n(\mathbf{C})$ is not a Hopf algebra

In classifying semisimple Hopf algebras, a major problem of interest is to find the precise conditions for a semisimple algebra in order to carry a Hopf algebra stucture. In this paragraph we make a start off in solving this problem by excluding quite a big class of semisimple algebras from the game.
We denote the linear space of right H-comodule homomorphisms from V to W by $\mathrm{ComHom}(V, W)$.

Theorem *Let H be a Hopf algebrawith $\iota^2 = id$, and U_R, V_R, W_R right H-comodules. Then the linear map*

$$\Phi : Hom_C(U_R, V_R \otimes W_R) \longrightarrow Hom_C(V_R^*, W_R \otimes U_R^*) : v \otimes w \otimes u^* \longmapsto w \otimes u^* \otimes v$$

gives an isomorphism between $\mathrm{ComHom}(U_R, V_R \otimes W_R)$ *and* $\mathrm{ComHom}(V_R^*, W_R \otimes U_R^*)$.

Proof (The maps we use are denoted as follows: $\psi : U_R \longrightarrow U_R \otimes H : u \longmapsto \sum u_0 \otimes u_1$, $\tilde{\psi}: U_L^* \longrightarrow H \otimes U_L^* : u^* \longmapsto \sum u_1^* \otimes u_0^*$, $\psi' : U_R^* \longrightarrow U_R^* \otimes H : u^* \longmapsto \sum u_0^* \otimes \iota(u_1^*)$)

I) First we are going to look what it means for an $f \in \mathrm{Hom}(U_R, V_R)$ to be in $\mathrm{ComHom}(U_R, V_R)$.

For every $u^* \in U_L^*$ (left comodule) and $u \in U_R$ we have:

$$\sum u_0^*(u)u_1^* = \sum u^*(u_0)u_1.$$

Write $f = \sum v \otimes u^*$. Now $f \in \mathrm{ComHom}(U_R, V_R)$ if and only if $\psi_V \circ f = (f \otimes id) \circ \psi_U$.
As we have seen already for every $u \in U$ we have:

$$\psi_V \circ f(u) = \sum v_0 \otimes u^*(u)v_1,$$

$$(f \otimes id) \circ \psi_U(u) = \sum f(u_0) \otimes u_1 = \sum u^*(u_0)v \otimes u_1 = \sum v \otimes u_0^*(u)u_1^*.$$

So: $f \in \mathrm{ComHom}(U_R, V_R)$ if and only if $\sum v_0 \otimes v_1 \otimes u^* = \sum v \otimes u_1^* \otimes u_0^*$

Now we can write down the criteria we are looking for:

$$f \in \mathrm{ComHom}(U_R, V_R \otimes W_R) \Leftrightarrow \sum v_0 \otimes w_0 \otimes v_1 w_1 \otimes u^* = \sum v \otimes w \otimes u_1^* \otimes u_0^*$$

and

$$\Phi(f) \in \mathrm{ComHom}(V_R^*, W_R \otimes U_R^*) \Leftrightarrow \sum w_0 \otimes u_0^* \otimes w_1 \iota(u_1^*) \otimes v = \sum w \otimes u^* \otimes \iota(v_1) \otimes v_0$$

(Note that in the last equation $\iota(v_1)$ is written (instead of v_1) because V^* is considered as a RIGHT comodule here).

II)"$\Rightarrow$" Suppose $f \in \mathrm{ComHom}(U_R, V_R \otimes W_R)$. Then:

$$\begin{array}{ccccc}
\sum v \otimes w \otimes \iota(u_1^*) \otimes u_0^* & = & \sum v_0 \otimes w_0 \otimes \iota(v_1 w_1) \otimes u^* & & \\
\downarrow{\scriptstyle id\otimes\rho\otimes id\otimes id} & & \downarrow{\scriptstyle id\otimes\rho\otimes id\otimes id} & & \\
\sum v \otimes w_0 \otimes w_1 \otimes \iota(u_1^*) \otimes u_0^* & = & \sum v_0 \otimes w_0 \otimes w_1 \otimes \iota(v_1 w_2) \otimes u^* & & \\
\downarrow{\scriptstyle (id\otimes id\otimes m\otimes id)} & & \downarrow{\scriptstyle (id\otimes id\otimes m\otimes id)} & & \\
\sum v \otimes w_0 \otimes w_1 \iota(u_1^*) \otimes u_0^* & = & \sum v_0 \otimes w_0 \otimes w_1 \iota(w_2)\iota(v_1) \otimes u_* & = & \sum v_0 \otimes w_0 \otimes \epsilon(w_1)\iota(v_1) \otimes u^*
\end{array}$$

The right-hand-side equals $\sum v_0 \otimes w \otimes \iota(v_1) \otimes u^*$, and hence $\Phi(f) \in \mathrm{ComHom}(V_R^*, W_R \otimes U_R^*)$.

"$\Leftarrow$" Suppose that $\Phi(f) \in \text{ComHom}(V_R^*, W_R \otimes U_R^*)$. Then:

$$\begin{array}{ccccc}
\sum w \otimes u^* \otimes \iota(v_1) \otimes v_0 & = & \sum w_0 \otimes u_0^* \otimes w_1 \iota(u_1^*) \otimes v & & \\
\downarrow {\scriptstyle id\otimes id\otimes\iota\otimes id} & & \downarrow {\scriptstyle id\otimes id\otimes\iota\otimes id} & & \\
\sum w \otimes u^* \otimes v_1 \otimes v_0 & = & \sum w_0 \otimes u_0^* \otimes u_1^* \iota(w_1) \otimes v & & \\
\downarrow {\scriptstyle \rho\otimes id\otimes id\otimes id} & & \downarrow {\scriptstyle \rho\otimes id\otimes id\otimes id} & & \\
\sum w_0 \otimes w_1 \otimes u^* \otimes v_1 \otimes v_0 & = & \sum w_0 \otimes w_1 \otimes u_0^* \otimes u_1^* \iota(w_2) \otimes v & & \\
\downarrow {\scriptstyle (id\otimes m\otimes id\otimes id)\circ(id\otimes\tau\otimes id\otimes id)\circ(id\otimes id\otimes\tau\otimes id)} & & \downarrow {\scriptstyle (id\otimes m\otimes id\otimes id)\circ(id\otimes\tau id\otimes id)\circ(id\otimes id\otimes\tau\otimes id)} & & \\
\sum w_0 \otimes v_1 w_1 \otimes u^* \otimes v_0 & = & \sum w_0 \otimes u_1^* \iota(w_2) w_1 \otimes u_0^* \otimes v & = & \sum w_0 \otimes u_1^* \epsilon(w_1) \otimes u_0^* \otimes
\end{array}$$

The right-hand-side equals $\sum w \otimes u_1^* \otimes u_0^* \otimes v$, and hence $f \in \text{ComHom}(U_R, V_R \otimes W_R)$ □

A last remark: Let V, W be right H-comodules, with V simple, W cosemisimple. It is easy to see that, because C is algebraically closed, $\dim_C(\text{ComHom}(V, W))$ =the number of times V appears in the decomposition of W [See CR Prop 3.33].

Theorem *$C \oplus M_n(C)$ is not a Hopf algebra*

Proof Suppose it was. Then, as a coalgebra $H^* = C \cdot 1 \oplus C_2$ with C_2 a n^2-dimensional simple subcoalgebra. By the second lemma there exist (up to isomorphism) two simple right H-comodules: $C \cdot 1$ and V, say, with $\dim_C(V) = n$.

The right H-comodule $V \otimes V^*$ has dimension n^2. Because H is cosemisimple, $V \otimes V^*$ can be decomposed in a direct sum of simple subcomodules, i.e.

$$V \otimes V^* = k_1 C \oplus k_2 V$$

for some $k_1, k_2 \in Z_0$.

We know that the number of times $C \cdot 1$ is contained in $V \otimes V^*$ equals $\dim_C(\text{ComHom}(C \cdot 1, V \otimes V^*))$. The last theorem tells us that $\text{ComHom}(C \cdot 1, V \otimes V^*) \cong \text{ComHom}(V^*, V^*)$, and $\dim_C(\text{ComHom}(V^*, V^*)) = 1$

But then $n^2 = 1 + k_2 n$ which is impossible if $n > 1$ □

0.5 The algebra $\mathbf{C} \oplus M_n(\mathbf{C})$ is not a Hopf algebra (2)

Suppose it was. Then H^* would have exactly two irreducible characters, χ_1 and χ_2 say, (afforded by the simple left H-comodules V_1 and V_2 resp.), with $\deg(\chi_1) = 1$ and $\deg(\chi_2) = n$.

Now, for the character χ_2^2 (the character afforded by $V_2 \otimes V_2$), we have: $\deg(\chi_2^2) = n^2$. By Corollary 7 we know that $\chi_2^2 = r\chi_2 + (n^2 - rn)\chi_1$ for some $r \in Z_0$

Let D be the simple subcoalgebra associated with V_2, and let $\{a_{ij}\}$ be a basis for D with the property stated in Lemma 12. Because ι is a coalgebra antihomomorphism, $E := \iota(D)$ is also a simple subcoalgebra. Define $b_{ji} := \iota(a_{ij})$. E can be considered as a right H-comodule (with the comultiplication as comodule map). Choose some simple subcomodule F. Then E is the simple subcoalgebra associated with F, and $\{b_{ij}\}$ is a basis as introduced in Lemma 12. Now $\chi_F = \sum_i b_{ii} = \sum_i \iota(a_{ii}) = \iota(\sum_i a_{ii}) = \iota(\chi_2)$, which means that $\iota(\chi_2)$ is an irreducible character as well. Since $\dim(D)$ =$\dim(E)$, we know that χ_2 and $\iota(\chi_2)$ must be of the same degree, and thus we must have: $\iota(\chi_2) = \chi_2$.

Because $C_1 := C \cdot 1_{H^*}$ is the only one-dimensional subcoalgebra in H^*, any one-dimensional comodule V has C_1 as its associated simple subcoalgebra. Using Lemma 12 and Definition 16 we see that χ_1 is an element $a_{11} \in C_1$ with $\mu(a_{11}) = a_{11} \otimes a11$ and $\epsilon(a_{11}) = 1$ i.e. $\chi_1 = 1_{H^*}$. By Theorem 18 we now have:

$$\chi_2^2(\lambda) = (\chi_2\iota(\chi_2))(\lambda) = 1,$$
$$\chi_2(\lambda) = \chi_2 \cdot 1_{H^*}(\lambda) = (\chi_2\iota(1_{H^*}))(\lambda) = (\chi_2\iota(\chi_1))(\lambda) = 0.$$

Thus $1 = \chi_2^2(\lambda) = (r\chi_2 + (n^2 - rn)\chi_1)(\lambda) = n^2 - rn = n(n-r)$. If $n > 1$ this is impossible and we are done. $\square$

0.6 References

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